Calculus Application Problem - Let It Hang!

1. The Problem: There are two functions which occur in many advanced applications of calculus and engineering. One of these applications occurs when a chain, a cable, or a power line is strung between two poles. This shape looks like a parabola, but is actually a **catenary**. We will explore these functions, look at some of their properties, their applications, and rewrite them in their Maclaurin Series representations.

2. Looking at the equation for a catenary.

A catenary can be expressed as a combination of the two functions e^x and e^{-x}. This combination can be written

as: $y = \frac{1}{2}(e^x + e^{-x})$

Sketch this function on your calculator with a window of [-4 , 4] by [-2 , 10] to see the basic shape of the function.

3. Finding an equation that fits a chain.

a. On a large sheet of graph paper, draw a horizontal x-axis near the bottom of the paper and a vertical y-axis down the middle of the paper.

b. Hang a chain over the paper by taping it to the upper corners of your paper. Hang it so the vertex of the chain is above the x-axis. (Is the vertex of the chain on the y-axis? It needs to be!))

c. By measuring (using the squares on the paper), find the x and y coordinates for the vertex and for the two endpoints of your chain. Try to measure accurate to the nearest 0.1 unit.

Vertex: (______ , ______) **Left endpt:** (______ , ______) **Right endpt:** (______ , ______)

d. The general equation for a hanging chain is $y = \frac{k}{2}(e^{\frac{x}{k}} + e^{-\frac{x}{k}}) + C$, where the value of k is determined by the tension in the chain and the weight of the chain.

e. Substitute the ordered pairs for the vertex and the right endpt for x and y into the general equation for the catenary to obtain two equations in terms of C and k. Simplify where obvious.

1. ____________________ 2. ____________________

f. Solve both equations for C.

C = ____________________ C = ____________________

g. Set the two equations equal to each other and use your calculator to solve for k graphically. Then find the value of C by substitution. Round answers to the nearest 0.001.

k = __________ C = __________

h. The equation of the chain is: y = ______________________________

To check your results, graph the equation of your chain on your graphing calculator. What would be a good choice for your WINDOW? (Remember the piece of paper?)

Xmin ________ Xmax ________ Ymin ________ Ymax ________

(Note: If the graph does not look like the chain, you may want to select 5:ZoomSqr from the ZOOM menu.)

By TRACING, check to see if the vertex and endpts are on the function. (If not, find out what is wrong!)

i. Carefully lift the chain above the paper and tape it to the wall without removing the taped ends. (You are going to rehang the chain in the same position later).

j. Complete the table below by calculating the y-values for the given x-values of your function. Round to the nearest one decimal place.

X	±2	±4	±6	±8	±10
Y					

k. Plot these points on your paper by measuring as accurately as you can. Label your points.

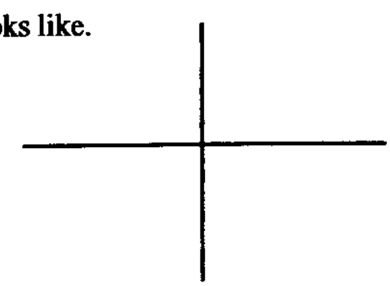
l. Rehang the chain. How closely does your calculated data fit the shape of the actual chain? (Show your instructor!)

m. Remove the chain from the wall, stretch it out straight, and measure it's length. We will use the length of the chain later in this aplication problem.

Measured value = ______________________.

4. Defining the Hyperbolic Trigonometric Functions

Let's also consider the function formed by combining e^x and e^{-x} with the rule: $y = \frac{1}{2}(e^x + e^{-x})$. Sketch the graph of this function to see what it looks like.



Since the properties of these two functions, $y=\frac{1}{2}(e^x+e^{-x})$ and $y=\frac{1}{2}(e^x-e^{-x})$, are very similar to the properties associated with sinx and cosx, and they have a similar relationship with a hyperbola that the trig functions have with circle, these functions are called <u>hyperbolic trigonometric functions</u> and are defined as follows:

Hyperbolic cosine of x: $\cosh x=\frac{1}{2}(e^x+e^{-x})$

Hyperbolic sine of x: $\sinh x=\frac{1}{2}(e^x-e^{-x})$

Find the "cosh" and "sinh" functions (in the CATALOG of your calculator) and sketch their graphs with their corresponding exponential forms to verify that you get the same graphs.

5. Properties of the Hyperbolic Trigonometric Functions

As was stated earlier, many of the properties of the hyperbolic trig functions are very similar to the properties of the trig functions. Below are properties of trigonometric functions (that we should be familiar with), followed by a property of a hyberbolic trig function. On engineering paper, use the <u>exponential form</u> of the hyperbolic functions to prove the stated property of the hyperbolic function. Be sure to show all of the steps in your proofs!

a. Trigonometric: $\cos^2 x+\sin^2 x=1$

Hyperbolic: $\cosh^2 x-\sinh^2 x=1$

b. Trigonometric: $\sin 2x=2\sin x\cos x$

Hyperbolic: $\sinh 2x=2\sinh x\cosh x$

c. Trigonometric: $\frac{d}{dx}(\sin x)=\cos x$ and $\frac{d}{dx}(\cos x)=-\sin x$

Hyperbolic: $\frac{d}{dx}(\sinh x)=\cosh x$ and $\frac{d}{dx}(\cosh x)=\sinh x$

The hyperbolic functions also have inverse functions associated with them. It can be shown that the "inverse hyperbolic sine function" is defined as:

$\sinh^{-1} x=\ln\left(x+\sqrt{x^2+1}\right)$ (You don't have to show this!)

d. Trigonometric: $\frac{d}{dx}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^2}}$

Hyperbolic: Find the derivative $\frac{d}{dx}\left(\sinh^{-1} x\right)$ and simplify it. Your final result should be very similar to the derivative of the $\sin^{-1} x$ above!

6. Maclaurin Series Representations of the Hyperbolic Functions

In this section we will find the Maclaurin Series for the two functions $y = \cosh x$ and $y = \sinh x$ by using the ɩponential forms and the properties that we know that apply to Maclaurin/Taylor series in general. (Do al of the rollowing on another sheet of paper.)

a. Write the first 5 terms and the sigma notation that we have found for the series representation for e^x.

b. Use the series for e^x and write the first 5 terms and the sigma notation for the series representation for e^{-x}.

c. Use the two series above and write the first 5 terms and the sigma notation for the series representation for $e^x + e^{-x}$.

d. Write the first 5 terms and the sigma notation for the series representation for $\frac{1}{2}(e^x + e^{-x})$.

e. Since $\cosh x = \frac{1}{2}(e^x + e^{-x})$, the answer above is the Maclaurin series for coshx. How does this compare to the Maclaurin series for the trig function cosx?

f. Using properties of Maclaurin series discussed in class, write the first 5 terms and the sigma notation for the series representation for sinhx. Explain what you are doing to produce this series.

7. Finding the Length of the Chain

The general form of the catenary $y = \frac{k}{2}(e^{\frac{x}{k}} + e^{-\frac{x}{k}}) + C$ can be written in the "cosh" form as $y = k \cdot \cosh\left(\frac{x}{k}\right) + C$.

Rewrite the specific equation of the chain found during recitation in its "cosh" form.

$y =$ ______________________________

Graph it to check that it is equivalent to the exponential form.

We derived the following formula to determine the **actual length L of a function** on an interval [a , b].

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

a. Find y' for the equation of the chain and simplify it.

$y' =$ ______________________________ = ______________________

b. The expression **$1+(y')^2$** can be simplified with one of the identities you proved earlier. Rewrite the expression using this identity, and take the square root of it.

c. Write and evaluate (with your calculator) the integral expression that can be used to calculate the **length** of your chain between the **right endpt** and **left endpt**.

$L =$ ______________________________ = ______________

How does this value compare to the length of the chain when it was measured? ______________